4

The vibration of continuous structures

Continuous structures such as beams, rods, cables and plates can be modelled by discrete mass and stiffness parameters and analysed as multi-degree of freedom systems, but such a model is not sufficiently accurate for most purposes. Furthermore, mass and elasticity cannot always be separated in models of real systems. Thus mass and elasticity have to be considered as distributed or continuous parameters.

For the analysis of structures with distributed mass and elasticity it is necessary to assume a homogeneous, isotropic material that follows Hooke's law.

Generally, free vibration is the sum of the *principal modes*. However, in the unlikely event of the elastic curve of the body in which motion is excited coinciding exactly with one of the principal modes, only that mode will be excited. In most continuous structures the rapid damping out of high-frequency modes often leads to the fundamental mode predominating.

4.1 LONGITUDINAL VIBRATION OF A THIN UNIFORM BEAM

Consider the longitudinal vibration of a thin uniform beam of cross-sectional area S, material density ρ, and modulus E under an axial force P, as shown in Fig. 4.1.

The net force acting on the element is $P + \partial P/\partial x \,.\, \mathrm{d}x - P$, and this is equal to the product of the mass of the element and its acceleration.

From Fig. 4.1,

$$\frac{\partial P}{\partial x}\,\mathrm{d}x = \rho\, S\, \mathrm{d}x\,\frac{\partial^2 u}{\partial t^2}.$$

Now strain $\partial u/\partial x = P/SE$, so

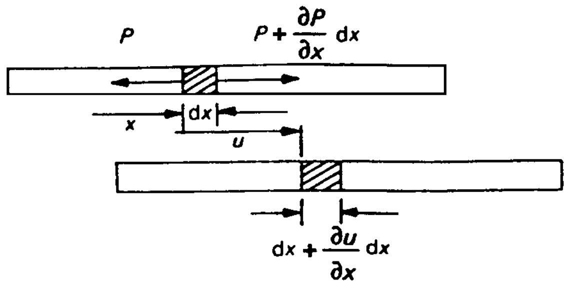


Fig. 4.1. Longitudinal beam vibration.

$$\partial P/\partial x = SE(\partial^2 u/\partial x^2).$$

Thus

$$\partial^2 u/\partial t^2 = (E/\rho)(\partial^2 u/\partial x^2),$$

or

$$\partial^2 u/\partial x^2 = (1/c^2)(\partial^2 u/\partial t^2), \quad \text{where } c = \sqrt{(E/\rho)}.$$

This is the wave equation. The velocity of propagation of the displacement or stress wave in the bar is c.

The wave equation

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2}\right)\left(\frac{\partial^2 u}{\partial t^2}\right)$$

can be solved by the method of separation of variables and assuming a solution of the form

$$u(x, t) = F(x)G(t).$$

Substituting this solution into the wave equation gives

$$\frac{\partial^2 F(x)}{\partial x^2} G(t) = \frac{1}{c^2}\frac{\partial^2 G(t)}{\partial t^2} F(x),$$

that is

$$\frac{1}{F(x)}\frac{\partial^2 F(x)}{\partial x^2} = \frac{1}{c^2}\frac{1}{G(t)}\frac{\partial^2 G(t)}{\partial t^2}$$

The LHS is a function of x only, and the RHS is a function of t only, so partial derivatives are no longer required. Each side must be a constant, $-(\omega/c)^2$ say. (This quantity is chosen for convenience of solution.) Then

$$\frac{d^2 F(x)}{dx^2} + \left(\frac{\omega}{c}\right)^2 F(x) = 0$$

and

$$\frac{d^2G(t)}{dt^2} + \omega^2 G(t) = 0.$$

Hence

$$F(x) = A \sin\left(\frac{\omega}{c}\right)x + B \cos\left(\frac{\omega}{c}\right)x$$

and

$$G(t) = C \sin \omega t + D \cos \omega t.$$

The constants A and B depend upon the boundary conditions, and C and D upon the initial conditions. The complete solution to the wave equation is therefore

$$u = \left(A \sin\left(\frac{\omega}{c}\right)x + B \cos\left(\frac{\omega}{c}\right)x\right)\left(C \sin \omega t + D \cos \omega t\right).$$

Example 29

Find the natural frequencies and mode shapes of longitudinal vibrations for a free–free beam with initial displacement zero.

Since the beam has free ends, $\partial u/\partial x = 0$ at $x = 0$ and $x = l$. Now

$$\frac{\partial u}{\partial x} = \left(A\left(\frac{\omega}{c}\right)\cos\left(\frac{\omega}{c}\right)x - B\left(\frac{\omega}{c}\right)\sin\left(\frac{\omega}{c}\right)x\right)\left(C \sin \omega t + D \cos \omega t\right).$$

Hence

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = A\left(\frac{\omega}{c}\right)(C \sin \omega t + D \cos \omega t) = 0, \quad \text{so that } A = 0$$

and

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = \left(\frac{\omega}{c}\right)\left(-B \sin\left(\frac{\omega l}{c}\right)\right)\left(C \sin \omega t + D \cos \omega t\right) = 0.$$

Thus $\sin(\omega l/c) = 0$, since B $\neq$ 0, and therefore

$$\frac{\omega l}{c} = \frac{\omega l}{\sqrt{(E/\rho)}} = \pi, 2\pi, \ldots, n\pi, \ldots,$$

that is,

$$\omega_n = \frac{n\pi}{l}\sqrt{\left(\frac{E}{\rho}\right)} \text{ rad/s},$$

where $\omega = c/\text{wavelength}$. These are the natural frequencies.

If the initial displacement is zero, $D = 0$ and

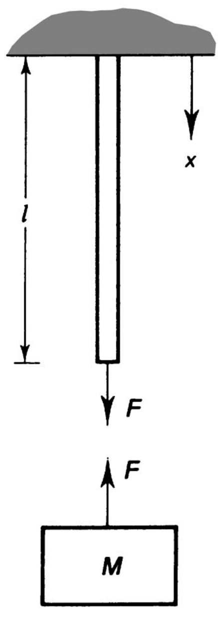
$$u_n = B' \cos\left(\frac{n\pi}{l}\right)x \,.\, \sin\left(\frac{n\pi}{l}\right)\sqrt{\left(\frac{E}{\rho}\right)}\, t.$$

where $B' = B \times C$. Hence the mode shape is determined.

Example 30

A uniform vertical rod of length l and cross-section S is fixed at the upper end and is loaded with a body of mass M on the other. Show that the natural frequencies of longitudinal vibration are determined by

$$\omega l\sqrt{(\rho/E)} \tan \omega l\sqrt{(\rho/E)} = S\rho l/M.$$



At $x = 0$, $u = 0$, and at $x = l$, $F = SE\,(\partial u/\partial x)$.

Also

$$F = SE\,(\partial u/\partial x) = -M(\partial^2 u/\partial t^2).$$

The general solution is

$$u = (A \sin(\omega/c)x + B\cos(\omega/c)x)(C \sin \omega t + D \cos \omega t).$$

Now, $u_{x=o} = 0$, so $B = 0$,

thus

$$u = (A \sin(\omega/c)x)(C \sin \omega t + D \cos \omega t),$$

$$(\partial u/\partial x)_{x=l} = (A(\omega/c)\cos(\omega l/c))(C\sin\omega t + D\cos\omega t)$$

and

$$(\partial^2 u/\partial t^2)_{x=l} = (-A\omega^2\sin(\omega l/c))(C\sin\omega t + D\cos\omega t),$$

so

$$F = SEA\,(\omega/c)\cos(\omega l/c)(C\sin\omega t + D\cos\omega t)$$

$$= MA\omega^2\sin(\omega l/c)(C\sin\omega t + D\cos\omega t).$$

Hence $(\omega l/c)\tan(\omega l/c) = SlE/Mc^2$, and

$$\omega l\sqrt{(\rho/E)}\tan\omega l\sqrt{(\rho/E)} = S\rho l/M, \quad \text{since } c^2 = E/\rho.$$

4.2 TRANSVERSE VIBRATION OF A THIN UNIFORM BEAM

The transverse or lateral vibration of a thin uniform beam is another vibration problem in which both elasticity and mass are distributed. Consider the moments and forces acting on the element of the beam shown in Fig. 4.2. The beam has a cross-sectional area A, flexural rigidity EI, material of density ρ and Q is the shear force.

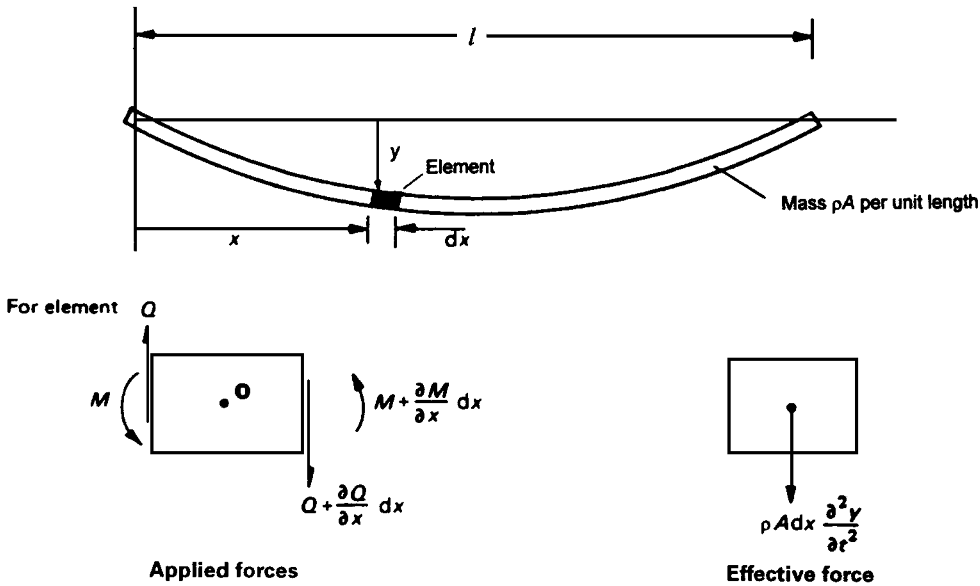


Fig. 4.2. Transverse beam vibration.

Then for the element, neglecting rotary inertia and shear of the element, taking moments about O gives

$$M + Q\frac{\mathrm{d}x}{2} + Q\frac{\mathrm{d}x}{2} + \frac{\partial Q}{\partial x}\mathrm{d}x\frac{\mathrm{d}x}{2} = M + \frac{\partial M}{\partial x}\mathrm{d}x,$$

that is,

$$Q = \partial M/\partial x.$$

Summing forces in the y direction gives

$$\frac{\partial Q}{\partial x}\,\mathrm{d}x = \rho A\,\mathrm{d}x\,\frac{\partial^2 y}{\partial t^2}.$$

Hence

$$\frac{\partial^2 M}{\partial x^2} = \rho A\,\frac{\partial^2 y}{\partial t^2}.$$

Now EI is a constant for a prismatical beam, so

$$M = -EI\,\frac{\partial^2 y}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 M}{\partial x^2} = -EI\,\frac{\partial^4 y}{\partial x^4}.$$

Thus

$$\frac{\partial^4 y}{\partial x^4} + \left(\frac{\rho A}{EI}\right)\frac{\partial^2 y}{\partial t^2} = 0.$$

This is the general equation for the transverse vibration of a uniform beam.

When a beam performs a normal mode of vibration the deflection at any point of the beam varies harmonically with time, and can be written

$$y = X\,(B_1 \sin \omega t + B_2 \cos \omega t),$$

where X is a function of x which defines the beam shape of the normal mode of vibration. Hence

$$\frac{\mathrm{d}^4 X}{\mathrm{d}x^4} = \left(\frac{\rho A}{EI}\right)\omega^2 X = \lambda^4 X,$$

where

$\lambda^4 = \rho A \omega^2/EI$. This is the *beam equation.*

The general solution to the beam equation is

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x + C_3 \cosh \lambda x + C_4 \sinh \lambda x,$$

where the constants $C_{1,2,3,4}$ are determined from the boundary conditions.

For example, consider the transverse vibration of a thin prismatical beam of length l, simply supported at each end. The deflection and bending moment are therefore zero at each end, so that the boundary conditions are $X = 0$ and $\mathrm{d}^2X/\mathrm{d}x^2 = 0$ at $x = 0$ and $x = l$.

Substituting these boundary conditions into the general solution above gives

at $x = 0$, $X = 0$; thus $0 = C_1 + C_3$, and

$$\text{at } x = 0, \frac{d^2X}{dx^2} = 0; \quad \text{thus } 0 = C_1 - C_3;$$

that is,

$$C_1 = C_3 = 0 \quad \text{and} \quad X = C_2 \sin \lambda x + C_4 \sinh \lambda x.$$

Now

$$\text{at } x = l, X = 0 \quad \text{so that} \quad 0 = C_2 \sin \lambda l + C_4 \sinh \lambda l,$$

and

$$\text{at } x = l, \frac{d^2X}{dx^2} = 0, \quad \text{so that} \quad 0 = C_2 \sin \lambda l - C_4 \sinh \lambda l;$$

that is,

$$C_2 \sin \lambda l = C_4 \sinh \lambda l = 0.$$

Since $\lambda l \neq 0$, $\sinh \lambda l \neq 0$ and therefore $C_4 = 0$.

Also $C_2 \sin \lambda l = 0$. Since $C_2 \neq 0$ otherwise $X = 0$ for all x, then $\sin \lambda l = 0$. Hence $X = C_2 \sin \lambda x$ and the solutions to $\sin \lambda l = 0$ give the natural frequencies. These are

$$\lambda = 0, \frac{\pi}{l}, \frac{2\pi}{l}, \frac{3\pi}{l}, \ldots$$

so that

$$\omega = 0, \left(\frac{\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)}, \left(\frac{2\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)}, \left(\frac{3\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)}, \ldots \text{ rad/s};$$

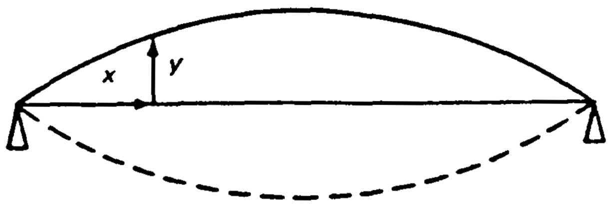
$\lambda = 0$, $\omega = 0$ is a trivial solution because the beam is at rest, so the lowest or first natural frequency is $\omega_1 = (\pi/l)^2\sqrt{(EI/A\rho)}$ rad/s, and the corresponding mode shape is $X = C_2 \sin \pi x/l$; this is the first mode; $\omega_2 = (2\pi/l)^2\sqrt{(EI/A\rho)}$ rad/s is the second natural frequency, and the second mode is $X = C_2 \sin 2\pi x/l$, and so on. The mode shapes are drawn in Fig. 4.3.

These sinusoidal vibrations can be superimposed so that any initial conditions can be represented. Other end conditions give frequency equations with the solution where the values of α are given in Table 4.1.

$$\omega = \frac{\alpha}{l^2}\sqrt{\left(\frac{EI}{A\rho}\right)} \text{ rad/s},$$

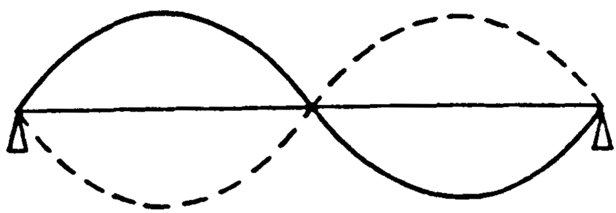
1st mode shape, one half-wave:

$$y = C_2 \sin \pi \left(\frac{x}{l}\right)(B_1 \sin \omega_1 t + B_2 \cos \omega_1 t); \quad \omega_1 = \left(\frac{\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)} \text{ rad/s.}$$



2nd mode shape, two half-waves:

$$y = C_2 \sin 2\pi \left(\frac{x}{l}\right)(B_1 \sin \omega_2 t + B_2 \cos \omega_2 t); \quad \omega_2 = \left(\frac{2\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)} \text{ rad/s.}$$



3rd mode shape, three half-waves:

$$y = C_2 \sin 3\pi \left(\frac{x}{l}\right)(B_1 \sin \omega_3 t + B_2 \cos \omega_3 t); \quad \omega_3 = \left(\frac{3\pi}{l}\right)^2 \sqrt{\left(\frac{EI}{A\rho}\right)} \text{ rad/s.}$$

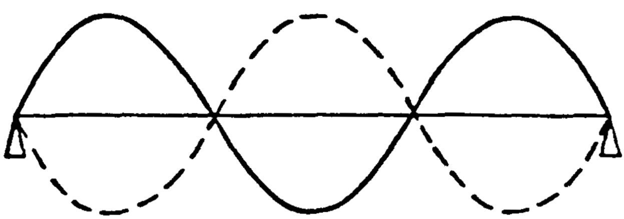


Fig. 4.3. Transverse beam vibration mode shapes and frequencies.

Table 4.1

End conditions	Frequency equation	1st mode	2nd mode	3rd mode	4th mode	5th mode
Clamped–free	$\cos \lambda l \cosh \lambda l = -1$	3.52	22.4	61.7	21.0	199.9
Pinned–pinned	$\sin \lambda l = 0$	9.87	39.5	88.9	157.9	246.8
Clamped–pinned	$\tan \lambda l = \tanh \lambda l$	15.4	50.0	104.0	178.3	272.0
Clamped–clamped or Free-free	$\cos \lambda l \cosh \lambda l = 1$	22.4	61.7	121.0	199.9	298.6

The natural frequencies and mode shapes of a wide range of beams and structures are given in *Formulas for Natural Frequency and Mode Shape* by R. D. Blevins (Van Nostrand, 1979).

4.2.1 The whirling of shafts

An important application of the theory for transverse beam vibration is to the whirling of shafts. If the speed of rotation of a shaft is increased, certain speeds will be reached at which violent instability occurs. These are the critical speeds of whirling. Since the loading on the shaft is due to centrifugal effects the equation of motion is exactly the same as for transverse beam vibration. The centrifugal effects occur because it is impossible to make the centre of mass of any section coincide exactly with the axis of rotation, because of a lack of homogeneity in the material and other practical difficulties.

Example 31

A uniform steel shaft which is carried in long bearings at each end has an effective unsupported length of 3 m. Calculate the first two whirling speeds.

Take $I/A = 0.1 \times 10^{-3}$ m^2, $E = 200$ GN/m^2, and $\rho = 8000$ kg/m^3.

Since the shaft is supported in long bearings, it can be considered to be 'built in' at each end so that, from Table 4.1,

$$\omega = \frac{\alpha}{l^2}\sqrt{\left(\frac{EI}{A\rho}\right)} \text{ rad/s},$$

where $\alpha_1 = 22.4$ and $\alpha_2 = 61.7$. For the shaft,

$$\sqrt{\left(\frac{EI}{A\rho}\right)} = \sqrt{\left(\frac{200 \times 10^9 \times 0.1 \times 10^{-3}}{8000}\right)} = 50 \text{ m}^2/\text{s},$$

so that the first two whirling speeds are:

$$\omega_1 = \frac{22.4}{9} 50 = 124.4 \text{ rad/s},$$

so

$$f_1 = \frac{\omega_1}{2\pi} = \frac{124.4}{2\pi} = 19.8 \text{ cycle/s and } N_1 = 1188 \text{ rev/min}$$

and

$$N_2 = \frac{61.7}{22.4} 1188 = 3272 \text{ rev/min.}$$

Rotating this shaft at speeds at or near to the above will excite severe resonance vibration.

4.2.2 Rotary inertia and shear effects

When a beam is subjected to lateral vibration so that the depth of the beam is a significant proportion of the distance between two adjacent nodes, rotary inertia of beam elements and transverse shear deformation arising from the severe contortions of the beam during vibration make significant contributions to the lateral deflection. Therefore rotary inertia and shear effects must be taken into account in the analysis of high-frequency vibration of all beams, and in all analyses of deep beams.

The moment equation can be modified to take into account rotary inertia by a term $\rho I\, \partial^3 y/(\partial x\, \partial t^2)$, so that the beam equation becomes

$$EI\frac{\partial^4 y}{\partial x^4} - \rho I\frac{\partial^3 y}{\partial x \partial t^2} + \rho A\frac{\partial^2 y}{\partial t^2} = 0.$$

Shear deformation effects can be included by adding a term

$$\frac{EI\rho}{kg}\frac{\partial^4 y}{\partial x^2\, \partial t^2},$$

where k is a constant whose value depends upon the cross section of the beam. Generally, k is about 0.85. The beam equation then becomes

$$EI\frac{\partial^4 y}{\partial x^4} - \frac{EI\rho}{kg}\frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho A\frac{\partial^2 y}{\partial t^2} = 0.$$

Solutions to these equations are available, which generally lead to a frequency a few percent more accurate than the solution to the simple beam equation. However, in most cases the modelling errors exceed this. In general, the correction due to shear is larger than the correction due to rotary inertia.

4.2.3 The effect of axial loading

Beams are often subjected to an axial load, and this can have a significant effect on the lateral vibration of the beam. If an axial tension T exists, which is assumed to be constant for small-amplitude beam vibrations, the moment equation can be modified by including a term $T\partial^2 y/\partial x^2$, so that the beam equation becomes

$$EI\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + \rho A\frac{\partial^2 y}{\partial t^2} = 0.$$

Tension in a beam will increase its stiffness and therefore increase its natural frequencies; compression will reduce these quantities.

Example 32

Find the first three natural frequencies of a steel bar 3 cm in diameter, which is simply supported at each end, and has a length of 1.5 m. Take $\rho = 7780\ \text{kg/m}^3$ and $E = 208\ \text{GN/m}^2$.

For the bar,

$$\sqrt{\left(\frac{EI}{A\rho}\right)} = \sqrt{\left(\frac{208 \times 10^9 \times \pi(0.03)^4/64}{\pi(0.03/2)^2\ 7780}\right)}\ \text{m/s}^2 = 38.8\ \text{m/s}^2.$$

Thus

$$\omega_1 = \frac{\pi^2}{1.5^2}\ 38.8 = 170.2\ \text{rad/s} \quad \text{and} \quad f_1 = 27.1\ \text{Hz}.$$

Hence

$$f_2 = 27.1 \times 4 = 108.4\ \text{Hz}$$

and

$$f_3 = 27.1 \times 9 = 243.8\ \text{Hz}.$$

If the beam is subjected to an axial tension T, the modified equation of motion leads to the following expression for the natural frequencies:

$$\omega_n^2 = \left(\frac{n\pi}{l}\right)^2 \frac{T}{A\rho} + \left(\frac{n\pi}{l}\right)^4 \frac{EI}{A\rho}.$$

For the case when $T = 1000$ N the correction to ω_1^2 is ω_c^2, where

$$\omega_c^2 = \left(\frac{\pi}{1.5}\right)^2\left(\frac{1000}{\pi(0.03/2)^2\ 7780}\right) = 795\ (\text{rad/s})^2.$$

That is, $f_c = 4.5$ Hz. Hence $f_1 = \sqrt{(4.5^2 + 27.1^2)} = 27.5$ Hz.

4.2.4 Transverse vibration of a beam with discrete bodies

In those cases where it is required to find the lowest frequency of transverse vibration of a beam that carries discrete bodies, Dunkerley's method may be used. This is a simple analytical technique which enables a wide range of vibration problems to be solved using a hand calculator. Dunkerley's method uses the following equation:

$$\frac{1}{\omega_1^2} \simeq \frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} + \frac{1}{P_4^2} + \ldots,$$

where ω_1 is the lowest natural frequency of a system and P_1, P_2, P_3, ... are the frequencies of each body acting alone (see section 3.2.1.2).

Example 33

A steel shaft (ρ = 8000 kg/m^3, E = 210 GN/m^2) 0.055 m diameter, running in self-aligning bearings 1.25 m apart, carries a rotor of mass 70 kg, 0.4 m from one bearing. Estimate the lowest critical speed.

For the shaft alone

$$\sqrt{\left(\frac{EI}{A\rho}\right)} = \sqrt{\left(\frac{210 \times 10^9 \times \pi(0.055)^4/64}{\pi(0.055/2)^2\ 8000}\right)} = 70.45 \text{ m/s}^2.$$

$$\text{Thus } P_1 = \left(\frac{\pi}{1.25}\right)^2 70.45 = 445 \text{ rad/s} = 4249 \text{ rev/min.}$$

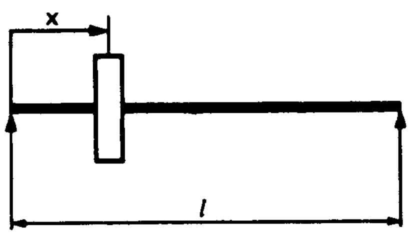
This is the lowest critical speed for the shaft without the rotor. For the rotor alone, neglecting the mass of the shaft,

$$P_2 = \sqrt{(k/m)} \text{ rad/s}$$

and

$$k = 3EIl/(x^2(l - x)^2),$$

where x = 0.4 m and l = 1.25 m.



Thus

$$k = 3.06 \text{ MN/m}$$

and

$$P_2 = \sqrt{((3.06 \times 10^6)/70)}$$
$$= 209.1 \text{ rad/s} = 1996 \text{ rev/min.}$$

Now using Dunkerley's method,

$$1/N_1^2 = 1/4249^2 + 1/1996^2, \text{ hence } N_1 = 1807 \text{ rev/min.}$$

4.2.5 Receptance analysis

Many structures can be considered to consist of a number of beams fastened together. Thus if the receptances of each beam are known, the frequency equation of the structure can easily be found by carrying out a subsystem analysis (section 3.2.3). The required receptances can be found by inserting the appropriate boundary conditions in the general solution to the beam equation.

It will be appreciated that this method of analysis is ideal for computer solutions because of its repetitive nature.

For example, consider a beam that is pinned at one end ($x = 0$) and free at the other end ($x = l$). This type of beam is not commonly used in practice, but it is useful for analysis purposes. With a harmonic moment of amplitude M applied to the pinned end,

at $x = 0$, $X = 0$ (zero deflection) and

$$\frac{d^2X}{dx^2} = \frac{M}{EI} \text{ (bending moment } M\text{)},$$

and at $x = l$,

$$\frac{d^2X}{dx^2} = 0 \text{ (zero bending moment)}$$

and

$$\frac{d^3X}{dx^3} = 0 \text{ (zero shear force)}.$$

Now, in general,

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x + C_3 \cosh \lambda x + C_4 \sinh \lambda x.$$

Thus applying these boundary conditions,

$$0 = C_1 + C_3 \quad \text{and} \quad \frac{M}{EI} = -C_1\lambda^2 + C_3\lambda^2.$$

Also

$$0 = -C_1\lambda^2 \cos \lambda l - C_2\lambda^2 \sin \lambda l + C_3\lambda^2 \cosh \lambda l + C_4\lambda^2 \sinh \lambda l.$$

and

$$0 = C_1\lambda^3 \sin \lambda l - C_2\lambda^3 \cos \lambda l + C_3\lambda^3 \sinh \lambda l + C_4\lambda^3 \cosh \lambda l.$$

By solving these four equations $C_{1,2,3,4}$ can be found and substituted into the general solution. It is found that the receptance moment/slope at the pinned end is

$$\frac{(1 + \cos \lambda l \cosh \lambda l)}{EI\lambda (\cos \lambda l \sinh \lambda l - \sin \lambda l \cosh \lambda l)}$$

and at the free end is

$$\frac{2 \cos \lambda l \cosh \lambda l}{EI\lambda (\cos \lambda l \sinh \lambda l - \sin \lambda l \cosh \lambda l)}.$$

The frequency equation is given by

$$\cos \lambda l \sinh \lambda l - \sin \lambda l \cosh \lambda l = 0,$$

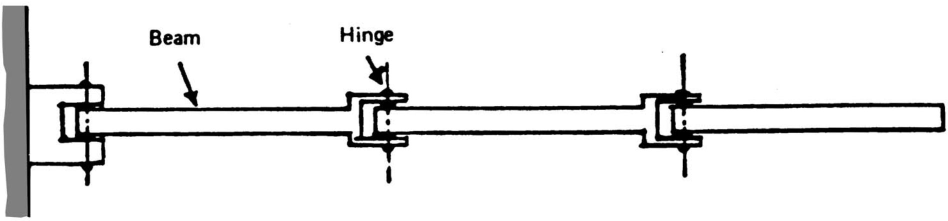
that is, $\tan \lambda l = \tanh \lambda l$.

Moment/deflection receptances can also be found.

By inserting the appropriate boundary conditions into the general solution, the receptance due to a harmonic moment applied at the free end, and harmonic forces applied to either end, can be deduced. Receptances for beams with all end conditions are tabulated in *The Mechanics of Vibration* by R. E. D. Bishop & D. C. Johnson (CUP, 1960/79), thereby greatly increasing the ease of applying this technique.

Example 34

A hinged beam structure is modelled by the array shown below:



The hinges are pivots with torsional stiffness k_T and their mass is negligible. All hinges and beams are the same.

It is required to find the natural frequencies of free vibration of the array, so that the excitation of these frequencies, and therefore resonance, can be avoided.

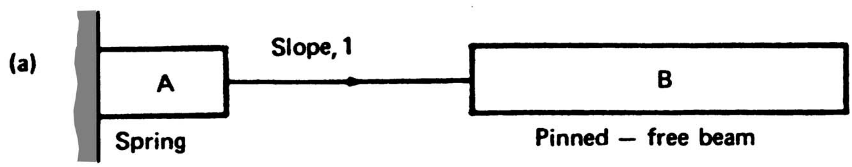
Since all the beams are identical, the receptance technique is relevant for finding the frequency equation. This is because the receptances of each subsystem are the same, which leads to some simplification in the analysis.

There are two approaches:

(i) to split the array into subsystems comprising torsional springs and beams,
(ii) to split the array into subsystems comprising spring–beam assemblies.

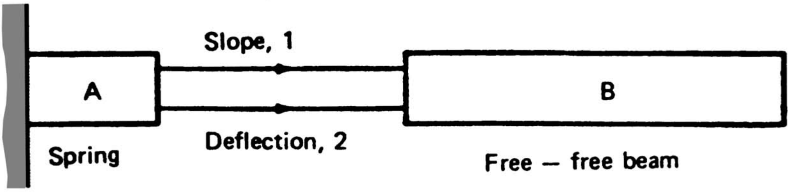
This approach results in a smaller number of subsystems.

Considering the first approach, and only the first element of the array, the subsystems could be either



or

(b)



For (a) the frequency equation is $\alpha_{11} + \beta_{11} = 0$, whereas for (b) the frequency equation is

$$\begin{vmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} \end{vmatrix} = 0,$$

where α_{11} is the moment/slope receptance for A, β_{11} is the moment/slope receptance for B, β_{12} is the moment/deflection receptance for B, β_{22} is the force/deflection receptance for B, and so on.

For (a), either calculating the beam receptances as above, or obtaining them from tables, the frequency equation is

$$\frac{1}{k_T} + \frac{\cos \lambda l \cosh \lambda l + 1}{EI\lambda(\cos \lambda l \sinh \lambda l - \sin \lambda l \cosh \lambda l)} = 0,$$

where

$$\lambda = \sqrt[4]{\left(\frac{A\rho\omega^2}{EI}\right)}.$$

For (b), the frequency equation is

$$\begin{vmatrix} \dfrac{1}{k_T} + \dfrac{\cos \lambda l \sinh \lambda l + \sin \lambda l \cosh \lambda l}{EI\lambda(\cos \lambda l \cosh \lambda l - 1)} & \dfrac{-\sin \lambda l \sinh \lambda l}{EI\lambda^2(\cos \lambda l \cosh \lambda l - 1)} \\ \dfrac{-\sin \lambda l \sinh \lambda l}{EI\lambda^2(\cos \lambda l \cosh \lambda l - 1)} & \dfrac{-(\cos \lambda l \sinh \lambda l - \sin \lambda l \cosh \lambda l)}{EI\lambda^3(\cos \lambda l \cosh \lambda l - 1)} \end{vmatrix} = 0,$$

which reduces to the equation given by method (a).

The frequency equation has to be solved after inserting the structural parameters, to yield the natural frequencies of the structure.

For the whole array it is preferable to use approach (ii), because this results in a smaller number of subsystems than (i), with a consequent simplification of the frequency equation. However, it will be necessary to calculate the receptances of the spring pinned–free beam if approach (ii) is adopted.

The analysis of structures such as frameworks can also be accomplished by the receptance technique, by dividing the framework to be analysed into beam substructures. For example, if the in-plane natural frequencies of a portal frame are required, it can be

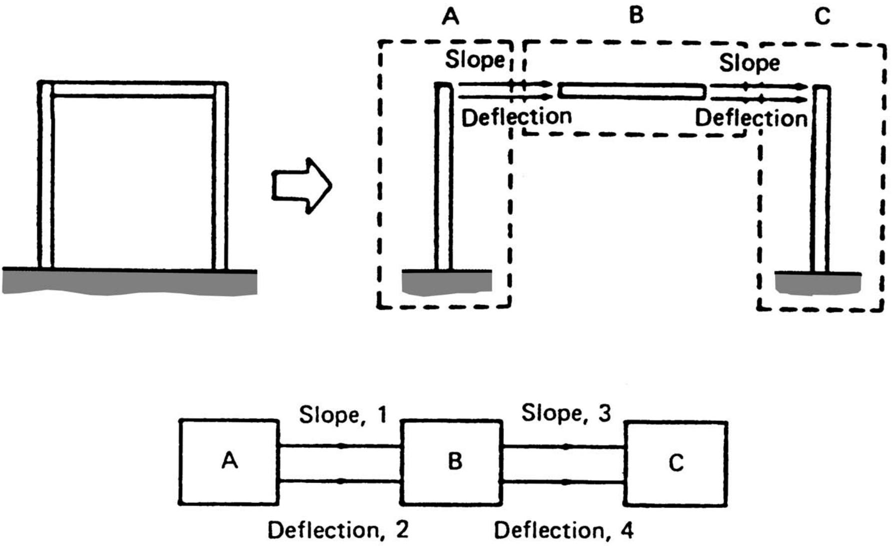


Fig. 4.4. Portal frame substructure analysis.

divided into three substructures coupled by the conditions of compatibility and equilibrium, as shown in Fig. 4.4.

Substructures A and C are cantilever beams undergoing transverse vibration, whereas B is a free–free beam undergoing transverse vibration. Beam B is assumed rigid in the horizontal direction, and the longitudinal deflection of beams A and C is assumed to be negligible.

Because the horizontal member B has no coupling between its horizontal and flexural motion $\beta_{12} = \beta_{14} = \beta_{23} = \beta_{34} = 0$, so that the frequency equation becomes

$$\begin{vmatrix} \alpha_{11} + \beta_{11} & \alpha_{11} & \beta_{13} & 0 \\ \alpha_{21} & \alpha_{22} + \beta_{22} & 0 & \beta_{24} \\ \beta_{31} & 0 & \gamma_{33} + \beta_{33} & \beta_{34} \\ 0 & \beta_{42} & \gamma_{43} & \gamma_{44} + \beta_{44} \end{vmatrix} = 0.$$

4.3 THE ANALYSIS OF CONTINUOUS STRUCTURES BY RAYLEIGH'S ENERGY METHOD

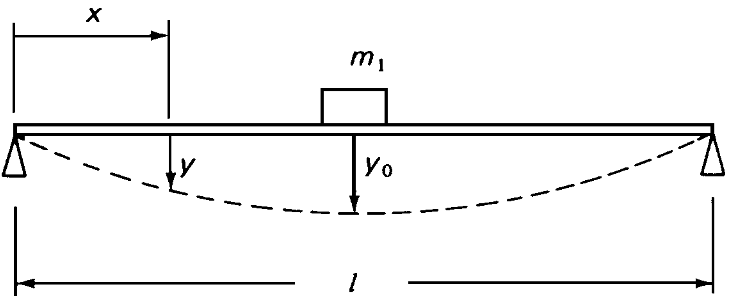
Rayleigh's method, as described in section 2.1.4, gives the lowest natural frequency of transverse beam vibration as

$$\omega^2 = \frac{\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int y^2\, dm}.$$

A function of x representing y can be determined from the static deflected shape of the beam, or a suitable part sinusoid can be assumed, as shown in the following examples.

Example 35

A simply supported beam of length l and mass m_2 carries a body of mass m_1 at its mid-point. Find the lowest natural frequency of transverse vibration.



This example has been fully discussed above (Example 4, p. 25). However, the Dunkerley method can also be used. Here

$$P_1^2 = \frac{48\,EI}{m_1 l^3} \text{ and } P_2^2 = \frac{EI\,\pi^4}{m_2 l^3}.$$

Thus

$$\frac{1}{\omega^2} = \frac{m_1 l^3}{48\,EI} + \frac{m_2 l^3}{\pi^4\,EI}.$$

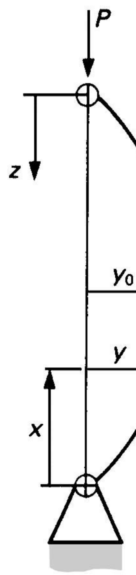
Hence

$$\omega^2 = \frac{EI\left(\dfrac{\pi}{l}\right)^4 \dfrac{l}{2}}{\left(1.015 m_1 + \dfrac{m_2}{2}\right)},$$

which is very close to the value determined by the Rayleigh method.

Example 36

A pin-ended strut of length l has a vertical axial load P applied. Determine the frequency of free transverse vibration of the strut, and the maximum value of P for stability. The strut has a mass m and a second moment of area I, and is made from material with modulus of elasticity E.



The deflected shape can be expressed by

$$y = y_0 \sin \pi \frac{x}{l},$$

since this function satisfies the boundary conditions of zero deflection and bending moment at $x = 0$ and $x = l$.

Now,

$$V_{\max} = \frac{1}{2} \int EI \left(\frac{d^2 y}{dx^2} \right)^2 dx - Pz,$$

where

$$\frac{1}{2} \int EI \left(\frac{d^2 y}{dx^2} \right)^2 dx = \frac{1}{2} \int_0^l EI \left(\frac{\pi}{l} \right)^4 y_0^2 \sin^2 \pi \frac{x}{l} \, dx$$

$$= \frac{EI}{4} \frac{\pi^4}{l^3} y_0^2,$$

and

$$z = \int_0^l \left(\sqrt{\left(1 + \left(\frac{dy}{dx} \right)^2 \right)} - 1 \right) dx$$

$$= \int_0^l \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$= \frac{1}{2}\int_0^l {y_0}^2 \left(\frac{\pi}{l}\right)^2 \cos^2 \pi \frac{x}{l}\, dx$$

$$= \frac{{y_0}^2}{4} \frac{\pi^2}{l}.$$

Thus

$$V_{max} = \left(\frac{EI}{4}\frac{\pi^4}{l^3} - \frac{P}{4}\frac{\pi^2}{l}\right){y_0}^2.$$

Now,

$$T_{max} = \frac{1}{2}\int y^2\, dm = \frac{1}{2}\int_0^l y^2 \frac{m}{l}\, dx$$

$$= \frac{1}{2}\int_0^l {y_0}^2 \sin^2 \pi \frac{x}{l}\frac{m}{l}\, dx = \frac{m}{4}{y_0}^2.$$

Thus

$$\omega^2 = \frac{\left(\frac{EI}{4}\frac{\pi^4}{l^3} - \frac{P}{4}\frac{\pi^2}{l}\right)}{\frac{m}{4}},$$

and

$$f = \frac{1}{2}\sqrt{\left(\frac{EI\,(\pi/l)^2 - P}{ml}\right)}\ \text{Hz}.$$

From section 2.1.4, for stability

$$\frac{dV}{dy_0} = 0 \quad \text{and} \quad \frac{d^2V}{d{y_0}^2} > 0,$$

that is,

$$y_0 = 0 \text{ and } EI\frac{\pi^2}{l^2} > P;$$

$y_0 = 0$ is the equilibrium position about which vibration occurs, and $P < EI\,\pi^2/l^2$ is the necessary condition for stability. $EI\,\pi^2/l^2$ is known as the Euler buckling load.

4.4 TRANSVERSE VIBRATION OF THIN UNIFORM PLATES

Plates are frequently used as structural elements so that it is sometimes necessary to analyse plate vibration. The analysis considered will be restricted to the vibration of thin uniform flat plates. Non-uniform plates that occur in structures, for example, those which are ribbed or bent, may best be analysed by the finite element technique, although exact theory does exist for certain curved plates and shells.

The analysis of plate vibration represents a distinct increase in the complexity of vibration analysis, because it is necessary to consider vibration in two dimensions instead of the single-dimension analysis carried out hitherto. It is essentially therefore, an introduction to the analysis of the vibration of multi-dimensional structures.

Consider a thin uniform plate of an elastic, homogeneous isotropic material of thickness h, as shown in Fig. 4.5.

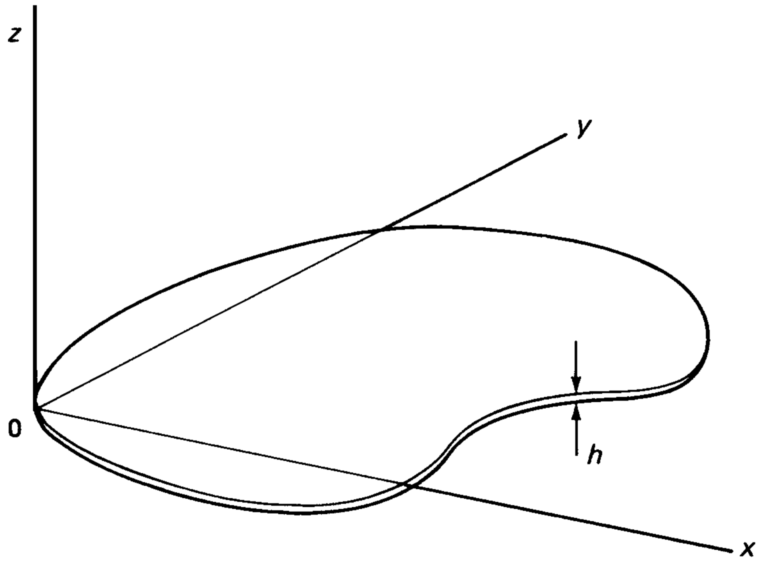


Fig. 4.5. Thin uniform plate.

If v is the deflection of the plate at a point (x, y), then it is shown in *Vibration Problems in Engineering* by S. Timoshenko (Van Nostrand, 1974), that the potential energy of bending of the plate is

$$\frac{D}{2}\iint\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 v}{\partial x.\partial y}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y$$

where the flexural rigidity,

$$D = \frac{Eh^3}{12(1 - \nu^2)}$$

and ν is Poisson's Ratio.

The kinetic energy of the vibrating plate is

$$\frac{\rho h}{2} \iint \dot{v}^2 \, dx \, dy,$$

where ρh is the mass per unit area of the plate.

In the case of a rectangular plate with sides of length a and b, and with simply supported edges, at a natural frequency ω, v can be represented by

$$v = \phi \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b},$$

where ϕ is a function of time.

Thus

$$V = \frac{\pi^4 ab}{8} D \phi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2$$

and

$$T = \frac{\rho h}{2} \frac{ab}{4} \dot{\phi}^2.$$

Since $d(T + V)/dt = 0$ in a conservative structure,

$$\frac{\rho h}{2} \frac{ab}{4} 2\dot{\phi}\ddot{\phi} + \frac{\pi^4 ab}{8} D \, 2\phi\dot{\phi} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 = 0;$$

that is, the equation of motion is

$$\rho h \ddot{\phi} + \pi^4 D \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \phi = 0.$$

Thus ϕ represents simple harmonic motion and

$$\phi = A \sin \omega_{mn} t + B \cos \omega_{mn} t,$$

where

$$\omega_{mn} = \pi^2 \sqrt{\left(\frac{D}{\rho h} \right)} \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} \right] \text{ rad/s}.$$

Now,

$$v = \phi \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b},$$

thus $v = 0$ when $\sin m\pi x/a = 0$ or $\sin n\pi y/b = 0$, and hence the plate has nodal lines when vibrating in its normal modes.

Typical nodal lines of the first six modes of vibration of a rectangular plate, simply supported on all edges, are shown in Fig. 4.6.

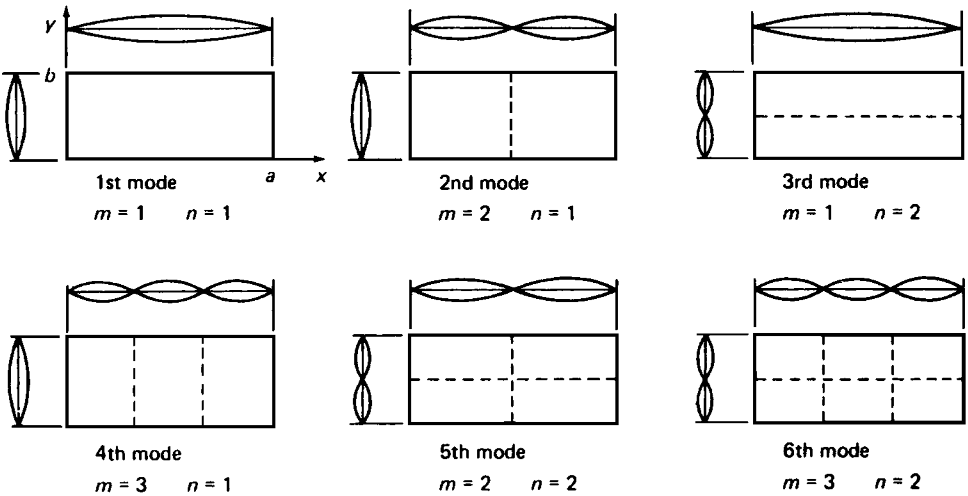


Fig. 4.6. Transverse plate vibration mode shapes.

An exact solution is only possible using this method if two opposite edges of the plate are simply supported: the other two edges can be free, hinged or clamped. If this is not the case, for example if the plate has all edges clamped, a series solution for v has to be adopted.

For a simply supported square plate of side $a(=b)$, the frequency of free vibration becomes

$$f = \pi \frac{m^2}{a^2} \sqrt{\left(\frac{D}{\rho h}\right)} \text{ Hz},$$

whereas for a square plate simply supported along two opposite edges and free on the others,

$$f = \frac{\alpha}{2\pi a^2} \sqrt{\left(\frac{D}{\rho h}\right)} \text{ Hz},$$

where $\alpha = 9.63$ in the first mode (1, 1), $\alpha = 16.1$ in the second mode (1, 2), and $\alpha = 36.7$ in the third mode (1, 3).

Thus the lowest, or fundamental, natural frequency of a simply supported/free square plate of side l and thickness d is

$$\frac{9.63}{2\pi l^2}\sqrt{\left(\frac{Ed^3}{12(1-\nu^2)\,\rho d}\right)} = \frac{10.09}{2\pi l^2}\sqrt{\left(\frac{Ed^2}{12\rho}\right)}\text{Hz},$$

if $\nu = 0.3$.

The theory for beam vibration gives the fundamental natural frequency of a beam simply supported at each end as

$$\frac{1}{2\pi}\left(\frac{\pi}{l}\right)^2\sqrt{\left(\frac{EI}{A\rho}\right)}\ \text{Hz}.$$

If the beam has a rectangular section $b \times d$, $I = \dfrac{bd^3}{12}$ and $A = bd$.

Thus

$$f = \frac{1}{2\pi}\left(\frac{\pi}{l}\right)^2\sqrt{\left(\frac{Ed^2}{12\rho}\right)}\ \text{Hz},$$

that is,

$$f = \frac{9.86}{2\pi l^2}\sqrt{\left(\frac{Ed^2}{12\rho}\right)}\ \text{Hz}.$$

This is very close (within about 2%) to the frequency predicted by the plate theory, although of course beam theory cannot be used to predict all the higher modes of plate vibration, because it assumes that the beam cross section is not distorted. Beam theory becomes more accurate as the aspect ratio of the beam, or plate, increases.

For a circular plate of radius a, clamped at its boundary, it has been shown that the natural frequencies of free vibration are given by

$$f = \frac{\alpha}{2\pi a^2}\sqrt{\left(\frac{D}{\rho h}\right)}\ \text{Hz},$$

where α is as given in Table 4.2.

Table 4.2

Number of nodal circles	Number of nodal diameters 0	1	2
0	10.21	21.26	34.88
1	39.77	60.82	84.58
2	89.1	120.08	153.81
3	158.18	199.06	242.71

The vibration of a wide range of plate shapes with various types of support is fully discussed in NASA publication SP-160 *Vibration of Plates* by A. W. Leissa.

4.5 THE FINITE ELEMENT METHOD

Many structures, such as a ship hull or engine crankcase, are too complicated to be analysed by classical techniques, so that an approximate method has to be used. It can be seen from the receptance analysis of complicated structures that breaking a dynamic structure down into a large number of substructures is a useful analytical technique, provided that sufficient computational facilities are available to solve the resulting equations. The finite element method of analysis extends this method to the consideration of continuous structures as a number of elements, connected to each other by conditions of compatibility and equilibrium. Complicated structures can thus be modelled as the aggregate of simpler structures.

The principal advantage of the finite element method is its generality; it can be used to calculate the natural frequencies and mode shapes of any linear elastic system. However, it is a numerical technique that requires a fairly large computer, and care has to be taken over the sensitivity of the computer output to small changes in input.

For beam type systems the finite element method is similar to the lumped mass method, because the system is considered to be a number of rigid mass elements of finite size connected by massless springs. The infinite number of degrees of freedom associated with a continuous system can thereby be reduced to a finite number of degrees of freedom, which can be examined individually.

The finite element method therefore consists of dividing the dynamic system into a series of elements by imaginary lines, and connecting the elements only at the intersections of these lines. These intersections are called nodes. It is unfortunate that the word node has been widely accepted for these intersections; this meaning should not be confused with the zero vibration regions referred to in vibration analysis. The stresses and strains in each element are then defined in terms of the displacements and forces at the nodes, and the mass of the elements is lumped at the nodes. A series of equations is thus produced for the displacement of the nodes and hence the system. By solving these equations the stresses, strains, natural frequencies and mode shapes of the system can be determined. The accuracy of the finite element method is greatest in the lower modes, and increases as the number of elements in the model increases. The finite element method of analysis is considered in *The Finite Element Method* by O. C. Zienkiewicz (McGraw Hill, 1977) and *A First Course in Finite Element Analysis* by Y. C. Pao (Allyn and Bacon, 1986).

4.6 THE VIBRATION OF BEAMS FABRICATED FROM MORE THAN ONE MATERIAL

Engineering structures are sometimes fabricated using composite materials. These applications are usually where high strength and low weight are required as, for example, in aircraft, space vehicles and racing cars. Composite materials are produced by embedding high-strength fibres in the form of filaments or yarn in a plastic, metal or ceramic matrix. They are more expensive than conventional materials but their application or manufacturing methods often justify their use.

The most common plastic materials used are polyester and epoxy resin, reinforced with glass. The glass may take the form of strands, fibres or woven fabrics. The desirable quality of glass fibres is their high tensile strength. Naturally the orientation and alignment or otherwise of the fibres can greatly affect the properties of the composite. Glass reinforced plastic (GRP) is used in such structures as boats, footbridges and car bodies. Boron fibres are more expensive than glass but because they are six times stiffer they are sometimes used in critical applications.

Carbon fibres are expensive, but they combine increased stiffness with a very high tensile strength, so that composites of carbon fibre and resin can have the same tensile strength as steel but weigh only a quarter as much. Because of this carbon fibre composites now compete directly with aluminium in many aircraft structural applications. Cost precludes its large-scale use, but in the case of the A320 Airbus, for example, over 850 kg of total weight is saved by using composite materials for control surfaces such as flaps, rudder, fin and elevators in addition to some fairings and structural parts.

Analysis of the vibration of such structural components can be conveniently carried out by the finite element method (section 4.5), or more usefully by the modal analysis method (section 3.3). However, composite materials are usually anisotropic so the analysis can be difficult. Inherent damping is often high however, even though it may be hard to predict due to variations in such factors as manufacturing techniques and fibre/matrix wetting.

Concrete is usually reinforced by steel rods, bars or mesh to contribute tensile strength. In reinforced concrete, the tensile strength of the steel supplements the compressive strength of the concrete to provide a structural member capable of withstanding high stresses of all kinds over large spans. It is a fairly cheap material and is widely used in the construction of bridges, buildings, boats, structural frameworks and roads.

It is sometimes appropriate, therefore, to fabricate structural components such as beams, plates and shells from more than one material, either in whole or in part, to take advantage of the different and supplementary properties of the two materials. Composites are also sometimes incorporated into highly stressed parts of a structure by applying patches of a composite to critical areas.

The vibration analysis of composite structures can be lengthy and difficult, but the fundamental frequency of vibration of a beam made from two materials can be determined using the energy principle, as follows.

Fig. 4.7 shows a cross section through a beam made from two materials 1 and 2 bonded at a common interface. Provided the bond is sufficiently good to prevent relative slip, a plane section before bending remains plane after bending so that the strain distribution is linear across the section, although the normal stress will change at the interface because of the difference in the elastic moduli of the two materials E_1 and E_2.

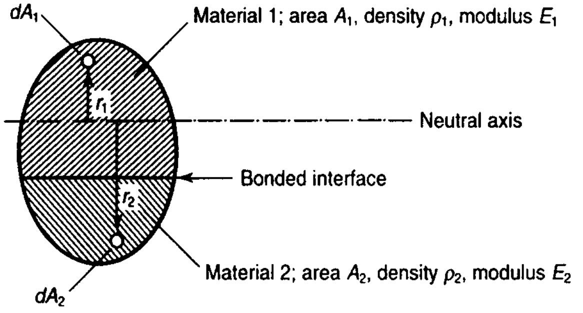


Fig. 4.7. Composite beam cross section.

Now, from section 2.1.4.2 and Fig. 2.11, the strain ε at a distance r from the neutral axis of a beam in bending is

$$\frac{(R + r)\mathrm{d}\theta - R\mathrm{d}\theta}{R\mathrm{d}\theta} = \frac{r}{R}.$$

Hence the strain at a distance r_1 from the neutral axis is

$$\varepsilon_1 = r_1 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2},$$

and similarly

$$\varepsilon_2 = r_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

Hence, the corresponding stresses are

$$\sigma_1 = E_1\varepsilon_1 = E_1 r_1 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$$

and

$$\sigma_2 = E_2\varepsilon_2 = E_2 r_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}.$$

The strain energy stored in the two materials per unit volume is $\mathrm{d}V_1 + \mathrm{d}V_2$

where

$$dV_1 = \frac{\sigma_1 \varepsilon_1}{2} = \frac{E_1}{2} r_1^2 \left(\frac{d^2y}{dx^2}\right)^2$$

and

$$dV_2 = \frac{\sigma_2 \varepsilon_2}{2} = \frac{E_2}{2} r_2^2 \left(\frac{d^2y}{dx^2}\right)^2.$$

Integrating over the volume of a beam of length l gives

$$V_{max} = \frac{E_1}{2} \int_0^l \int r_1^2 \left(\frac{d^2y}{dx^2}\right)^2 dA_1 \, dx + \frac{E_2}{2} \int_0^l \int r_2^2 \left(\frac{d^2y}{dx^2}\right)^2 dA_2 \, dx.$$

Now

$$I_1 = \int r_1^2 \, dA_1 \text{ and } I_2 = \int r_2^2 \, dA_2,$$

so

$$V_{max} = \left(\frac{E_1 I_1 + E_2 I_2}{2}\right) \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx.$$

I_1 and I_2 can only be calculated when the location of the neutral axis of the composite cross section is known. This can be found using an equivalent cross section for one material.

The mass per unit length of the composite is $\rho_1 A_1 + \rho_2 A_2$, so that

$$T_{max} = \left(\frac{\rho_1 A_1 + \rho_2 A_2}{2}\right) \int_0^l (y\omega)^2 \, dx.$$

A shape function has therefore to be assumed before T_{max} can be calculated.

Putting $T_{max} = V_{max}$ gives the natural frequency ω.

Example 37

A simply supported beam of length l is fabricated from two materials M1 and M2. Find the fundamental natural frequency of the beam using Rayleigh's method and the shape function

$$y = P \sin\left(\frac{\pi x}{l}\right).$$

$$V_{max} = \left(\frac{E_{M1} I_{M1} + E_{M2} I_{M2}}{2}\right) \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$= \left(\frac{E_{M1} I_{M1} + E_{M2} I_{M2}}{2}\right) \frac{P^2 \pi^4}{2l^3}$$

$$T_{\max} = \left(\frac{\rho_{M1} A_{M1} + \rho_{M2} A_{M2}}{2}\right) \int_0^l (y\omega)^2 \, dx$$

$$= \left(\frac{\rho_{M1} A_{M1} + \rho_{M2} A_{M2}}{2}\right) \frac{P^2 l}{2} \omega^2.$$

Putting $T_{\max} = V_{\max}$ gives

$$\omega^2 = \left(\frac{E_{M1} I_{M1} + E_{M2} I_{M2}}{\rho_{M1} A_{M1} + \rho_{M2} A_{M2}}\right) \frac{\pi^4}{l^4}$$

So that

$$\omega = \frac{\pi^2}{l^2} \sqrt{\left(\frac{E_{M1} I_{M1} + E_{M2} I_{M2}}{\rho_{M1} A_{M1} + \rho_{M2} I_{M2}}\right)} \text{ rad/s.}$$

I_{M1} and I_{M2} can be calculated once the position of the neutral axis has been found.

This method of analysis can obviously be extended to beams fabricated from more than two materials.